Analisi Matematica

Esercizi sui Limiti Notevoli – Soluzioni

Liceo Scientifico – Classi V – Prof. *Roberto Squellati*

1. $$\lim_{x\to\pi/4}\frac{\cos 2x}{\cos x-\cos(\pi/4)}=\lim_{x\to\pi/4}\frac{2\cos^2 x-1}{\cos x-\sqrt{2}/2}=\lim_{x\to\pi/4}\frac{2\left(\sqrt{2}\cos x-1\right)\left(\sqrt{2}\cos x+1\right)}{\sqrt{2}\left(\sqrt{2}\cos x-1\right)}=$$
$$=\lim_{x\to\pi/4}\sqrt{2}\left(\sqrt{2}\cos x+1\right)=\boxed{2\sqrt{2}}$$

2. $$\lim_{x\to\alpha}\frac{\operatorname{sen}(x-\alpha)}{\cos^2 x-\cos^2\alpha}=\lim_{t\to0}\frac{\operatorname{sen}t}{\cos^2(t+\alpha)-\cos^2\alpha}=\lim_{t\to0}\frac{\operatorname{sen}t}{[\cos(t+\alpha)-\cos\alpha]\cdot[\cos(t+\alpha)+\cos\alpha]}=$$
$$=\lim_{t\to0}\frac{2\operatorname{sen}\frac{t}{2}\cos\frac{t}{2}}{-2\cdot\operatorname{sen}\left(\frac{t+2\alpha}{2}\right)\cdot\operatorname{sen}\frac{t}{2}\cdot[\cos(t+\alpha)+\cos\alpha]}=\lim_{t\to0}\frac{\cos\frac{t}{2}}{-\operatorname{sen}\left(\frac{t}{2}+\alpha\right)\cdot[\cos(t+\alpha)+\cos\alpha]}=$$
$$=-\frac{1}{2\operatorname{sen}\alpha\cos\alpha}=\boxed{-\frac{1}{\operatorname{sen}2a}}$$

3. $$0\le\frac{\ln\left(2+\frac{1}{x}\right)}{x}\le\frac{\ln 3}{x}\Rightarrow\lim_{x\to+\infty}\frac{\ln\left(2+\frac{1}{x}\right)}{x}=\boxed{0}$$

4. $$\lim_{x\to+\infty}\frac{\ln\left(2x^2+3\right)}{\ln\left(x^3-1\right)}=\lim_{x\to+\infty}\frac{\ln\left[x^2\left(2+\frac{3}{x^2}\right)\right]}{\ln\left[x^3\left(1-\frac{1}{x^3}\right)\right]}=\lim_{x\to+\infty}\frac{\ln x^2+\ln\left(2+\frac{3}{x^2}\right)}{\ln x^3+\ln\left(1-\frac{1}{x^3}\right)}=$$
$$=\lim_{x\to+\infty}\frac{2\ln x+\ln\left(2+\frac{3}{x^2}\right)}{3\ln x+\ln\left(1-\frac{1}{x^3}\right)}=\lim_{x\to+\infty}\frac{2+\frac{\ln\left(2+3/x^2\right)}{\ln x}}{3+\frac{\ln\left(1-1/x^3\right)}{\ln x}}=\boxed{\frac{2}{3}}$$

5. $$\lim_{x\to+\infty}[\ln(1+e^x)-x]=\lim_{x\to+\infty}[\ln(1+e^x)-x\ln e]=\lim_{x\to+\infty}[\ln(1+e^x)-\ln e^x]=$$
$$=\lim_{x\to+\infty}\ln\left(\frac{1+e^x}{e^x}\right)=\lim_{x\to+\infty}\ln\left(1+\frac{1}{e^x}\right)=\boxed{0}$$

6. $$\lim_{x\to+\infty}\left(\frac{2x+1}{2x+3}\right)^{x-1}=\lim_{x\to+\infty}\left(\frac{1+\frac{1}{2x}}{1+\frac{3}{2x}}\right)^{x-1}=\lim_{x\to+\infty}\left[\left(\frac{1+\frac{1}{2x}}{1+\frac{3}{2x}}\right)^{x}\cdot\underbrace{\left(\frac{1+\frac{1}{2x}}{1+\frac{3}{2x}}\right)^{-1}}_{1}\right]=$$
$$=\lim_{x\to+\infty}\frac{\left(1+\frac{1}{2x}\right)^x}{\left(1+\frac{1}{2x/3}\right)^x}=\lim_{x\to+\infty}\frac{\left[\left(1+\frac{1}{2x}\right)^{2x}\right]^{1/2}}{\left[\left(1+\frac{1}{2x/3}\right)^{2x/3}\right]^{3/2}}=\frac{e^{1/2}}{e^{3/2}}=\boxed{e^{-1}}$$

7. $$\lim_{x\to+\infty}(x+1)^{-1/\ln x}=\lim_{x\to+\infty}\exp\left[-\frac{\ln(x+1)}{\ln x}\right]=\lim_{x\to+\infty}\exp\left\{-\frac{\ln[x(1+1/x)]}{\ln x}\right\}=$$
$$=\lim_{x\to+\infty}\exp\left\{-\frac{\ln x+\ln(1+1/x)}{\ln x}\right\}=\lim_{x\to+\infty}\exp\left\{-1-\frac{\ln(1+1/x)}{\ln x}\right\}=\boxed{e^{-1}}$$

8. $$\lim_{x\to+\infty}x^{1/\ln^2 x}=\lim_{x\to+\infty}\exp\left(\frac{\ln x}{\ln^2 x}\right)=\lim_{x\to+\infty}\exp\left(\frac{1}{\ln x}\right)=\boxed{1}$$

9. $$\lim_{x\to0}\frac{(1+2x)^4-1}{x}=\lim_{x\to0}\left[2\cdot\frac{(1+2x)^4-1}{2x}\right]=\lim_{t\to0}\left[2\cdot\frac{(1+t)^4-1}{t}\right]=\boxed{8}$$

10. $$\lim_{x\to1^+}\frac{e^{x-1}-1}{1-\cos(1-x)}=\lim_{x\to1^+}\frac{\dfrac{e^{x-1}-1}{x-1}\cdot(x-1)}{\dfrac{1-\cos(1-x)}{(1-x)^2}\cdot(1-x)^2}=\lim_{x\to1^+}\frac{2(x-1)}{(x-1)^2}=\lim_{x\to1^+}\frac{2}{x-1}=\boxed{+\infty}$$

11. $$\lim_{x\to0}\frac{1-\cos x}{\ln\left(1+\operatorname{tg}^2 x\right)}=\lim_{x\to0}\frac{\dfrac{1-\cos x}{x^2}\cdot x^2}{\dfrac{\ln\left(1+\operatorname{tg}^2 x\right)}{\operatorname{tg}^2 x}\cdot\operatorname{tg}^2 x}=\lim_{x\to0}\frac{x^2/2}{\dfrac{\operatorname{sen}^2 x}{\cos^2 x}}=\lim_{x\to0}\left[\frac{\cos^2 x}{2}\cdot\left(\frac{\operatorname{sen} x}{x}\right)^{-2}\right]=\boxed{\frac{1}{2}}$$

12. $$\lim_{x\to0}\frac{3^{\operatorname{sen} x}-1}{x}=\lim_{x\to0}\left(\frac{e^{\operatorname{sen} x\ln 3}-1}{\operatorname{sen} x\ln 3}\cdot\frac{\operatorname{sen} x}{x}\cdot\ln 3\right)=\boxed{\ln 3}$$

13. $$\lim_{x\to-1}\frac{1-\cos\left(x^2-1\right)}{e^{x+1}-1}=\lim_{x\to-1}\frac{\dfrac{1-\cos\left(x^2-1\right)}{\left(x^2-1\right)^2}\cdot\left(x^2-1\right)^2}{\dfrac{e^{x+1}-1}{x+1}\cdot(x+1)}=\lim_{x\to-1}\frac{(x-1)^2(x+1)^2}{2(x+1)}=$$
$$=\lim_{x\to+\infty}\frac{(x-1)^2(x+1)}{2}=\boxed{0}$$

14. $$\lim_{x\to+\infty}\frac{\operatorname{arctg} x-\dfrac{\pi}{2}}{x-\operatorname{sen} x}=\lim_{x\to+\infty}\frac{\operatorname{arctg} x-\dfrac{\pi}{2}}{x\cdot\left(1-\dfrac{\operatorname{sen} x}{x}\right)}=\boxed{0}$$

15. $$\lim_{x\to-\infty}\left(x+1+\sqrt{3x^2-5x-1}\right)=\lim_{x\to-\infty}\left(x+1+|x|\sqrt{3-\frac{5}{x}-\frac{1}{x^2}}\right)=$$
$$=\lim_{x\to-\infty}\left(x+1-x\sqrt{3-\frac{5}{x}-\frac{1}{x^2}}\right)=\lim_{x\to-\infty}x\left(1+\frac{1}{x}-\sqrt{3-\frac{5}{x}-\frac{1}{x^2}}\right)=\boxed{+\infty}$$

16. $$\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x=\lim_{x\to+\infty}\left(\frac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}\right)^x=\lim_{x\to+\infty}\frac{\left(1+\dfrac{1}{x}\right)^x}{\left[\left(1+\dfrac{1}{-x}\right)^{-x}\right]^{-1}}=\boxed{e^2}$$

17. $$\lim_{x\to\pi/2}\frac{3\operatorname{sen}^2 x+\operatorname{sen} x-4}{\cos x}=\lim_{x\to\pi/2}\frac{(3\operatorname{sen} x+4)(\operatorname{sen} x-1)}{\cos x}=\lim_{x\to\pi/2}\frac{-(3\operatorname{sen} x+4)\left(1-\operatorname{sen}^2 x\right)}{(1+\operatorname{sen} x)\cos x}=$$
$$=\lim_{x\to\pi/2}-\frac{(3\operatorname{sen} x+4)\cos^2 x}{(1+\operatorname{sen} x)\cos x}=\lim_{x\to\pi/2}-\frac{(3\operatorname{sen} x+4)\cos x}{1+\operatorname{sen} x}=\boxed{0}$$

18. $$\lim_{x\to\pi}\frac{\cos x+\cos 2x}{(x-\pi)^2}=\lim_{t\to0}\frac{\cos(\pi+t)+\cos(2\pi+2t)}{t^2}=\lim_{t\to0}\frac{-\cos t+\cos 2t}{t^2}=\lim_{t\to0}\frac{2\cos^2 t-\cos t-1}{t^2}=$$
$$=\lim_{t\to0}\frac{(2\cos t+1)(\cos t-1)}{t^2}=\lim_{t\to0}\left[-(2\cos t+1)\cdot\frac{1-\cos t}{t^2}\right]=\boxed{-\frac{3}{2}}$$

19. $$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{2x+1}=\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{-2x-1}=\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{-2x}\cdot\left(1+\frac{1}{x}\right)^{-1}\right]=$$
$$=\lim_{x\to\infty}\left\{\left[\left(1+\frac{1}{x}\right)^{x}\right]^{-2}\cdot\left(1+\frac{1}{x}\right)^{-1}\right\}=\boxed{e^{-2}}$$

20. $$\lim_{x\to0}\frac{(1-\cos x)\operatorname{sen}3x}{x^2\operatorname{sen}kx}=\lim_{x\to0}\left[3x\cdot\frac{\operatorname{sen}3x}{3x}\cdot\frac{1-\cos x}{x^2}\cdot\frac{1}{kx}\cdot\left(\frac{\operatorname{sen}kx}{kx}\right)^{-1}\right]=\lim_{x\to0}\frac{3x}{2kx}=\boxed{\frac{3}{2k}}$$

21. $$\lim_{x\to0^+}\frac{\operatorname{sen}\left(x^2+x\right)}{x^2}=\lim_{x\to0^+}\left[\frac{\operatorname{sen}\left(x^2+x\right)}{x^2+x}\cdot\frac{x^2+x}{x^2}\right]=\lim_{x\to0^+}\left(1+\frac{1}{x}\right)=\boxed{+\infty}$$

22. $$\lim_{x\to0}\frac{\left(1+x^2-x\right)^{\sqrt2}-1}{x}=\lim_{x\to0}\left[\frac{\left(1+x^2-x\right)^{\sqrt2}-1}{x^2-x}\cdot\frac{x^2-x}{x}\right]=\lim_{x\to0}\sqrt2\,(x-1)=\boxed{-\sqrt2}$$

23. $$\lim_{x\to\infty}\left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}=\lim_{x\to\infty}\left(\frac{3x+2-6}{3x+2}\right)^{\frac{x}{3}+\frac{1}{3}}=\lim_{x\to\infty}\left[\left(1-\frac{6}{3x+2}\right)^{\frac{x}{3}}\cdot\underbrace{\left(1-\frac{6}{3x+2}\right)^{\frac{1}{3}}}_{1}\right]=$$
$$=\lim_{x\to\infty}\left(1+\frac{1}{-\frac{3x+2}{6}}\right)^{\frac{x}{3}}=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{1}{3}\cdot\left(-\frac{3t+2}{6}\right)}=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{-\frac{2}{3}t-\frac{2}{9}}=$$
$$=\lim_{t\to\infty}\left\{\left[\left(1+\frac{1}{t}\right)^{t}\right]^{-2/3}\cdot\left(1+\frac{1}{t}\right)^{-2/9}\right\}=\boxed{e^{-2/3}}$$

24. $$\lim_{x\to1}\frac{\ln(7x-6)}{\ln(3x-2)}=\lim_{x\to1}\frac{\ln[7(x-1)+1]}{\ln[3(x-1)+1]}=\lim_{t\to0}\frac{\ln(1+7t)}{\ln(1+3t)}=\lim_{t\to0}\left[\frac{\ln(1+7t)}{7t}\cdot7t\cdot\frac{3t}{\ln(1+3t)}\cdot\frac{1}{3t}\right]=$$
$$=\lim_{t\to0}\frac{7}{3}\left[\frac{\ln(1+3t)}{3t}\right]^{-1}=\boxed{\frac{7}{3}}$$

25. $$\lim_{x\to0}\frac{e^{2x}-e^x}{\ln(1+2x)}=\lim_{x\to0}\left[e^x\cdot\frac{e^x-1}{x}\cdot x\frac{2x}{\ln(1+2x)}\cdot\frac{1}{2x}\right]=\lim_{x\to0}\left\{\frac{e^x}{2}\left[\frac{\ln(1+2x)}{2x}\right]^{-1}\right\}=\boxed{\frac{1}{2}}$$

26. $$\lim_{x\to4}\frac{4^{x-1}-64}{2\left(x^2-3x-4\right)}=\lim_{x\to4}\frac{4^{x-1}-4^3}{2(x-4)(x+1)}=\lim_{x\to4}\left[\frac{4^3}{2(x+1)}\cdot\frac{4^{x-1}-1}{x-4}\right]=\lim_{t\to0}\left(\frac{32}{t+5}\cdot\frac{4^t-1}{t}\right)=$$
$$=\frac{32\ln4}{5}=\boxed{\frac{64}{5}\ln2}$$

27. $$\lim_{x\to1}\frac{x^3-3x+2}{x^4-4x+3}=\lim_{x\to1}\frac{(x+2)(x-1)^2}{\left(x^2+2x+3\right)(x-1)^2}=\lim_{x\to1}\frac{x+2}{x^2+2x+3}=\boxed{\frac{1}{2}}$$

28. $$\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}=\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x\left(1+\frac{\sqrt{x+\sqrt{x}}}{x}\right)}}=\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x}\cdot\sqrt{1+\sqrt{\frac{x+\sqrt{x}}{x^2}}}}=$$
$$=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x\sqrt{x}}}}}=\boxed{1}$$

29. $$\lim_{x\to0}\frac{\operatorname{sen}5x}{\operatorname{sen}2x}=\lim_{x\to0}\left(5x\cdot\frac{\operatorname{sen}5x}{5x}\cdot\frac{2x}{\operatorname{sen}2x}\cdot\frac{1}{2x}\right)=\lim_{x\to0}\left[\frac{5}{2}\left(\frac{\operatorname{sen}2x}{2x}\right)^{-1}\right]=\boxed{\frac{5}{2}}$$

30. $\lim\limits_{x\to 1}\dfrac{\operatorname{sen}\pi x}{\operatorname{sen}3\pi x}=\lim\limits_{t\to 0}\dfrac{\operatorname{sen}\left[\pi\left(t+1\right)\right]}{\operatorname{sen}\left[3\pi\left(t+1\right)\right]}=\lim\limits_{t\to 0}\dfrac{\operatorname{sen}\left(\pi t+\pi\right)}{\operatorname{sen}\left[\left(3\pi t+\pi\right)+2\pi\right]}=\lim\limits_{t\to 0}\dfrac{-\operatorname{sen}\pi t}{\operatorname{sen}\left(3\pi t+\pi\right)}=\lim\limits_{t\to 0}\dfrac{-\operatorname{sen}\pi t}{-\operatorname{sen}3\pi t}=$

$$=\lim_{t\to 0}\left(\frac{\operatorname{sen}\pi t}{\pi t}\cdot\pi t\cdot\frac{3\pi t}{\operatorname{sen}3\pi t}\cdot\frac{1}{3\pi t}\right)=\lim_{t\to 0}\left[\frac{1}{3}\left(\frac{\operatorname{sen}3\pi t}{3\pi t}\right)^{-1}\right]=\boxed{\frac{1}{3}}$$

31. $\lim\limits_{x\to 0}\left(x\operatorname{sen}\dfrac{1}{x}\right)=\lim\limits_{x\to 0}\left[x\cdot\dfrac{\operatorname{sen}\left(1/x\right)}{x}\right]=\boxed{0}$

32. $\lim\limits_{x\to 0^+}\left(\ln x-\ln\operatorname{sen}2x\right)=\lim\limits_{x\to 0^+}\ln\dfrac{x}{\operatorname{sen}2x}=\lim\limits_{x\to 0^+}\ln\left(\dfrac{1}{2}\cdot\dfrac{2x}{\operatorname{sen}2x}\right)=\lim\limits_{x\to 0^+}\ln\left[\dfrac{1}{2}\left(\dfrac{\operatorname{sen}2x}{2x}\right)^{-1}\right]=\boxed{-\ln 2}$

33. $\lim\limits_{x\to 0}\left(\dfrac{\operatorname{sen}2x}{x}\right)^{x+1}=\lim\limits_{x\to 0}\left(2\cdot\dfrac{\operatorname{sen}2x}{2x}\right)^{x+1}=\boxed{2}$

34. $0\leq\sqrt{x}\left|\operatorname{sen}\ln x\right|\leq\sqrt{x}\Rightarrow\lim\limits_{x\to 0}\left(\sqrt{x}\operatorname{sen}\ln x\right)=\boxed{0}$

35. $\left|\dfrac{1}{x}\left(2+\operatorname{sen}\dfrac{\pi}{x}\right)\right|\geq\left|\dfrac{1}{x}\right|\Rightarrow\lim\limits_{x\to 0}\dfrac{1}{x}\left(2+\operatorname{sen}\dfrac{\pi}{x}\right)=\boxed{\infty}$

36. $\lim\limits_{x\to\alpha}\dfrac{\operatorname{sen}x-\operatorname{sen}\alpha}{x-\alpha}=\lim\limits_{t\to 0}\dfrac{\operatorname{sen}\left(t+\alpha\right)-\operatorname{sen}\alpha}{t}=\lim\limits_{t\to 0}\dfrac{2\cos\left(\dfrac{t+2\alpha}{2}\right)\operatorname{sen}\left(\dfrac{t}{2}\right)}{t}=$

$$=\lim_{t\to 0}\left[\frac{\operatorname{sen}\left(\dfrac{t}{2}\right)}{\left(\dfrac{t}{2}\right)}\cdot\cos\left(\frac{t+2\alpha}{2}\right)\right]=\boxed{\cos\alpha}$$

37. $\lim\limits_{x\to 0}\dfrac{x+\operatorname{sen}3x}{x-\operatorname{sen}2x}=\lim\limits_{x\to 0}\dfrac{x\left(1+\dfrac{\operatorname{sen}3x}{x}\right)}{x\left(1-\dfrac{\operatorname{sen}2x}{x}\right)}=\lim\limits_{x\to 0}\dfrac{1+3\cdot\dfrac{\operatorname{sen}3x}{3x}}{1-2\cdot\dfrac{\operatorname{sen}2x}{2x}}=\boxed{-4}$

38. $\lim\limits_{x\to\infty}\dfrac{x+\operatorname{sen}x}{x+\cos x}=\lim\limits_{x\to\infty}\dfrac{x\left(1+\dfrac{\operatorname{sen}x}{x}\right)}{x\left(1+\dfrac{\cos x}{x}\right)}=\lim\limits_{x\to\infty}\dfrac{1+\dfrac{\operatorname{sen}x}{x}}{1+\dfrac{\cos x}{x}}=\boxed{1}$

39. $\lim\limits_{x\to+\infty}\dfrac{\log^2 x+\sqrt[3]{\log x}-4}{3\log x-1}=\lim\limits_{x\to+\infty}\dfrac{\left(1+\dfrac{\sqrt[3]{\log x}}{\log^2 x}-\dfrac{4}{\log^2 x}\right)\log^2 x}{\left(3-\dfrac{1}{\log x}\right)\log x}=$

$$=\lim_{x\to+\infty}\frac{\left(1+\sqrt[3]{\dfrac{1}{\log^5 x}}-\dfrac{4}{\log^2 x}\right)\log x}{3-\dfrac{1}{\log x}}=\boxed{+\infty}$$